

Hopf Algebras with Positive Bases

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Abstract

We show that if a finite dimensional Hopf algebra over \mathbf{C} has a basis such that all the structure constants are non-negative, then the Hopf algebra must be given by a finite group G and a factorization $G = G_+ G_-$ into two subgroups. We also show that Hopf algebras in the category of finite sets with correspondences as morphisms are classified in the similar way. Our results can be used to explain some results in Hopf algebras from set-theoretical viewpoint.

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1 Introduction

Recently much progress has been made in the classification of finite dimensional Hopf algebras (see [EG] [Mas] [BDG] and references quoted there). The classification theorems are often established under the assumption that the dimensions of the Hopf algebras have few divisors.

In this work, we prove two classification theorems under different kinds of assumptions. The first theorem (Theorem 1) classifies finite dimensional Hopf algebras over \mathbf{C} that admit positive bases. A basis of a Hopf algebra over \mathbf{C} is called positive if all the structure coefficients under the basis are non-negative real numbers (see Section 2 for more precise definitions). This class of Hopf algebras is closed under tensor product, dual and the Drinfel'd double construction. Our theorem says that if a Hopf algebra has a positive basis, then it must be isomorphic to the bicrossproduct Hopf algebra induced from a finite group G with a unique factorization $G = G_+ G_-$ of two subgroups. The construction of this type of Hopf algebras can be found in [Ta] and [Mj2].

The second theorem (Theorem 3) classifies Hopf algebras in the correspondence category. In the monoidal category of finite sets, we may formally introduce the concept of Hopf algebra by imposing all the axioms in terms of commutative diagrams. Such a definition is just that of finite groups. The correspondence category under our consideration still has finite sets as objects. But a morphism from X to Y is a subset of $X \times Y$ instead of a map from X to Y . This is a monoidal category in which the concept of Hopf algebra can be similarly defined. Our theorem shows that a Hopf algebra in the correspondence category is still given by a finite group G with a unique factorization $G = G_+ G_-$ of two subgroups.

One of the motivations to consider positive based Hopf algebras is that there are many natural Hopf algebras with nearly positive bases. We call a basis nearly positive if all the structure coefficients except the antipode under the basis are non-negative real numbers. For example, the modified quantum group $U_q(g)$ in [Lu] has a nearly positive basis realized in terms certain varieties. Another example is the direct sum $\oplus_1^\infty R(S_n)$ and $\oplus_1^\infty R(GL_n(F_q))$ of complex representation rings [Ze], where positive bases are given by irreducible representations. The positivity of structural coefficients are due to the very ways of defining the multiplication and the comultiplication. See [Ka] for more examples. All the examples above are infinite dimensional Hopf algebras. In the finite dimensional case, we know no example of a nearly positive basis that is not a positive basis.

In a subsequent paper [LYZ], we classify positive quasi-triangular structures on positively based Hopf algebras, and discuss its relation to set-theoretical Yang-Baxter equations.

The paper is organized as follows: We first present the basic concepts and notations, and state the first classification theorem. In the next two sections, we give detailed proof

of the theorem. In the final section, we discuss the correspondence category and outline the proof of the second classification theorem.

2 Positively Based Hopf Algebras

Given a group G , the group algebra $\mathbb{C}G$ and the dual group algebra $(\mathbb{C}G)^*$ have G and G^* as positive bases in the sense that all the structure constants are nonnegative. More generally, we call a basis B of a Hopf algebra H *positive* if

1. The coordinates of the unit 1 are non-negative;
2. The coordinates of the counit ϵ (with respect to the dual basis B^*) are non-negative, i.e., $\epsilon(b) \geq 0$ for all $b \in B$;
3. For any $b_1, b_2 \in B$, the coordinates of $b_1 b_2$ are non-negative;
4. For any $b \in B$, the coordinates of $\Delta(b)$ (with respect to the tensor basis $B \otimes B$) are non-negative;
5. For any $b \in B$, the coordinates of the antipode $S(b)$ are non-negative.

We call a Hopf algebra *positively based* if it has a positive basis.

It is easy to see that the dual and the tensor of positively based Hopf algebras are still positively based. For the antipode S of a finite dimensional Hopf algebra, it is well-known that $S^N = 1$ for some positive integer N [R]. This implies that if B is a positive basis of H , then the coordinates of $S^{-1} = S^{N-1}$ with respect to B are still non-negative. Therefore $B^* \otimes B$ is a positive basis of the Drinfel'd double $D(H)$.

The prototypical example of positively based Hopf algebras is the bicrossproduct Hopf algebra $H(G; G_+, G_-)$ induced from a finite group G with a unique factorization $G = G_+ G_-$ of two subgroups (see [Ta] and [Mj2]). By $G = G_+ G_-$ we mean G_{\pm} are subgroups of G , such that any $g \in G$ can be written as $g = g_+ g_-$ for unique $g_+ \in G_+$ and $g_- \in G_-$. By taking inverse, any $g \in G$ can also be written as $g = \bar{g}_- \bar{g}_+$ for unique $\bar{g}_+ \in G_+$ and $\bar{g}_- \in G_-$. Therefore to any $g \in G$ is associated four uniquely determined elements

$$\alpha_+(g) = g_+, \quad \beta_+(g) = \bar{g}_+, \quad \alpha_-(g) = \bar{g}_-, \quad \beta_-(g) = g_-,$$

given by

$$g = g_+ g_- = \bar{g}_- \bar{g}_+. \tag{1}$$

It is then easy to see that

$$G_- \times G_+ \rightarrow G_+, \quad (\bar{g}_-, \bar{g}_+) \mapsto (\bar{g}_- \bar{g}_+)_+ = g_+, \tag{2}$$

$$G_- \times G_+ \rightarrow G_-, \quad (\bar{g}_-, \bar{g}_+) \mapsto (\bar{g}_- \bar{g}_+)_- = g_-. \quad (3)$$

$$G_+ \times G_- \rightarrow G_+, \quad (g_+, g_-) \mapsto \overline{(g_+ g_-)}_+ = \bar{g}_+, \quad (4)$$

$$G_+ \times G_- \rightarrow G_-, \quad (g_+, g_-) \mapsto \overline{(g_+ g_-)}_- = \bar{g}_- \quad (5)$$

are all group actions. Because of this, we also write

$$g_+ = \bar{g}^- \bar{g}_+, \quad g_- = \bar{g}_-^{g_+}, \quad \bar{g}_+ = g_+^{g_-}, \quad \bar{g}_- = g_-^{g_+}.$$

The third and the fourth actions satisfy the following matching relations

$$g_+(g_- h_-) = g_+ g_- (g_+^{g_-}) h_-, \quad (h_+ g_+)^{g_-} = h_+^{(g_+ g_-)} g_+^{g_-}. \quad (6)$$

It is well-known that given two actions between two groups G_\pm satisfying the matching relations, we may reconstruct G . Similarly, G can be reconstructed from the first and the second actions, which satisfy similar matching relations. We will not use the first and the second actions in the proof of the classification theorem.

The Hopf algebra $H(G; G_+, G_-)$ has basis $\{\{g\} : g \in G\} = G$, and operations are given by

$$\begin{aligned} \{g\}\{h\} &= \begin{cases} \{\bar{g}_- h\} = \{gh_-\} & \text{in case } \bar{g}_+ = h_+ \\ 0 & \text{in case } \bar{g}_+ \neq h_+ \end{cases} \\ \Delta\{g\} &= \sum_{h\bar{k}_+=g, \bar{k}_-=h_-} \{h\} \otimes \{k\} = \sum_{h_+ k=g, \bar{k}_-=h_-} \{h\} \otimes \{k\} \\ 1 &= \sum_{g_+ \in G_+} \{g_+\} \\ \epsilon\{g\} &= \begin{cases} 1 & \text{in case } g_+ = e \\ 0 & \text{in case } g_+ \neq e \end{cases} \\ S\{g\} &= \{g^{-1}\} \end{aligned}$$

The basis G is clearly a positive one. The following theorem shows that, up to rescaling, this is all the finite dimensional positively based Hopf algebras.

Theorem 1 *Given any finite dimensional Hopf algebra over \mathbf{C} with a positive basis B , we can always rescale B by some positive numbers, so that (H, B) is isomorphic to $(H(G; G_+, G_-), G)$ for a unique group G and a unique factorization $G = G_+ G_-$.*

A consequence of the theorem is that the Drinfel'd double $D(H(G; G_+, G_-))$ is also of the form $H(\tilde{G}; \tilde{G}_+, \tilde{G}_-)$ for some group \tilde{G} and factorization $\tilde{G} = \tilde{G}_+ \tilde{G}_-$. By tracing our proof of the theorem, we find

$$\begin{aligned} \tilde{G} &= G \times G, \\ \tilde{G}_+ &= \{(g_+, g_-) : g_+ \in G_+, g_- \in G_-\} \cong G_+ \times G_-, \\ \tilde{G}_- &= \{(g, g) : g \in G\} \cong G. \end{aligned}$$

3 Classification up to Rescaling

In this section, we prove the classification theorem 1 up to rescaling. In the next section, we work out the necessary rescaling.

For a positive basis B of H , we denote by $B^* = \{b^* : b \in B\}$ the dual basis of H^* . The basis B^* is still positive. We also have the bases $B \otimes B, B \otimes B^*, \dots$, of $H \otimes H, H \otimes H^*, \dots$. We say an element $x \in H$ *contains* a term $b \in B$ if the b -coordinate of x is not zero. We can say similar things in $H^*, H \otimes H, H \otimes H^*, \dots$.

Given subsets X and Y of a basis B , the notations $X^*, X \otimes Y, X \otimes Y^*$, etc., carry the obvious meaning as subsets of $B \otimes B, B \otimes B^*$, etc.. The meaning of the phrases like X -term, X^* -term, $X \otimes Y$ -term, $X \otimes Y^*$ -term, etc., are self-evident.

We will often make use of the duality between H and H^* , both being positively based. This allows us to turn a result about product into a result about coproduct, and vice versa. The guiding principal for doing this is the following.

Lemma 1 *The coefficient of $b_1^* \otimes b_2^*$ in $\Delta(b^*)$ is the same as the coefficient of b in $b_1 b_2$. In particular, $\Delta(b^*)$ contains $b_1^* \otimes b_2^*$ if and only if $b_1 b_2$ contains b . Similarly, we may interchange the role of H and H^* .*

Proof: Let $b_1 b_2 = \sum \mu_c c$, and $\Delta(b^*) = \sum_{c_1, c_2 \in B} \lambda_{c_1, c_2} c_1^* \otimes c_2^*$. Then

$$\mu_b = \langle b^*, b_1 b_2 \rangle = \langle \Delta(b^*), b_1 \otimes b_2 \rangle = \lambda_{b_1, b_2}.$$

□

After a suitable rescaling by positive numbers, we have

$$1 = d_1 + \dots + d_k,$$

where d_1, \dots, d_k are distinct elements of B .

Lemma 2 *$d_i d_j = \delta_{ij} d_i$, and $\mathbf{C}d_1 + \dots + \mathbf{C}d_k$ is a commutative Hopf subalgebra of H .*

Proof: From $d_i = d_i \cdot 1 = d_i d_1 + \dots + d_i d_k$ and positivity, $d_i d_j$ is a nonnegative multiple of d_i . Similarly, from $d_j = 1 \cdot d_j$, we see that $d_i d_j$ is a nonnegative multiple of d_j . Therefore $d_i d_j = 0$ when $i \neq j$. Substituting this into $d_i = d_i d_1 + \dots + d_i d_k$, we get $d_i = d_i d_i$.

From $\Delta(d_1) + \dots + \Delta(d_k) = \Delta(1) = 1 \otimes 1 = \sum_{i,j} d_i \otimes d_j$ and positivity, we see that $\mathbf{C}d_1 + \dots + \mathbf{C}d_k$ is closed under Δ .

From $S(d_1) + \dots + S(d_k) = S(1) = 1 = d_1 + \dots + d_k$ and positivity, we see that $\mathbf{C}d_1 + \dots + \mathbf{C}d_k$ is closed under S .

Therefore $\mathbf{C}d_1 + \dots + \mathbf{C}d_k$ is a Hopf subalgebra of H .

□

Commutative Hopf algebras have been well-classified. We have a group G_+ , such that $\mathbf{C}d_1 + \cdots + \mathbf{C}d_k$ is isomorphic to the dual $(\mathbf{C}G_+)^*$ of the group algebra. More precisely, we have

$$\{d_1, \dots, d_k\} = G_+^* = \{d_{g_+} : g_+ \in G_+\}$$

and

$$\begin{aligned} d_{g_+}d_{h_+} &= \delta_{g_+, h_+}d_{g_+}, & \Delta(d_{g_+}) &= \sum_{h_+k_+=g_+} d_{h_+} \otimes d_{k_+}, \\ S(d_{g_+}) &= d_{g_+}^{-1}, & 1 &= \sum_{g_+ \in G_+} d_{g_+}, & \epsilon(d_{g_+}) &= \delta_{g_+, e} \end{aligned}$$

Lemma 3 *For any $b \in B$, there are unique $g_+, \bar{g}_+ \in G_+$, such that*

$$d_{h_+}b = \delta_{g_+, h_+}b, \quad bd_{h_+} = \delta_{\bar{g}_+, h_+}b.$$

Proof: From $b = 1 \cdot b = \sum_{h_+ \in G_+} d_{h_+}b$ and positivity, we have $d_{h_+}b = \lambda_{h_+}b$ for some nonnegative number λ_{h_+} . Then from $d_{g_+}d_{h_+} = \delta_{g_+, h_+}d_{g_+}$, we see that $\lambda_{g_+}\lambda_{h_+} = \delta_{g_+, h_+}\lambda_{g_+}$. Therefore all except one λ_* vanish. Moreover, the nonvanishing λ_* must be 1. \square

Denote the unique elements in the last lemma by

$$\alpha_+(b) = g_+, \quad \beta_+(b) = \bar{g}_+.$$

It follows from the lemma that

$$\alpha_+(b) = g_+ \iff d_{g_+}b = b \iff d_{g_+}b \neq 0; \quad (7)$$

$$\beta_+(b) = \bar{g}_+ \iff bd_{\bar{g}_+} = b \iff bd_{\bar{g}_+} \neq 0. \quad (8)$$

We also denote

$$\begin{aligned} B_{g_+, \bar{g}_+} &= \{b \in B : \alpha_+(b) = g_+, \beta_+(b) = \bar{g}_+\}, \\ B_{g_+, \bullet} &= \{b \in B : \alpha_+(b) = g_+\}, \\ B_{\bullet, \bar{g}_+} &= \{b \in B : \beta_+(b) = \bar{g}_+\}. \end{aligned}$$

Then B is the disjoint union of all B_{g_+, \bar{g}_+} .

Lemma 4 *If b_1b_2 contains b , then*

$$\alpha_+(b_1) = \alpha_+(b), \quad \beta_+(b_2) = \beta_+(b). \quad (9)$$

If $\Delta(b)$ contains $b_1 \otimes b_2$, then

$$\alpha_+(b) = \alpha_+(b_1)\alpha_+(b_2), \quad \beta_+(b) = \beta_+(b_1)\beta_+(b_2). \quad (10)$$

Proof: Suppose $b_1 b_2$ contains in b , and denote $g = \alpha_+(b)$, $\bar{g} = \beta_+(b)$. Then $b_1 b_2 d_{\bar{g}_+}$ and $d_{g_+} b_1 b_2$ still contain $d_{g_+} b = b = b d_{\bar{g}_+}$. Therefore $b_2 d_{\bar{g}_+} \neq 0$ and $d_{g_+} b_1 \neq 0$. By (7) and (8), this implies (9).

Let $\Delta(b) = \sum_{b_1, b_2 \in B} \lambda_{b_1, b_2} b_1 \otimes b_2$. Then

$$\begin{aligned} \Delta(b) = \Delta(d_{g_+} b) &= \sum_{b_1, b_2 \in B, h_+ k_+ = g_+} \lambda_{b_1, b_2} d_{h_+} b_1 \otimes d_{k_+} b_2 \\ &= \sum_{\alpha_+(b_1) = h_+, \alpha_+(b_2) = k_+, h_+ k_+ = g_+} \lambda_{b_1, b_2} b_1 \otimes b_2. \end{aligned}$$

Therefore $\Delta(b)$ contains only the terms $b_1 \otimes b_2$ satisfying $\alpha_+(b_1) \alpha_+(b_2) = h_+ k_+ = g_+ = \alpha_+(b)$. The proof for $\beta_+(b) = \beta_+(b_1) \beta_+(b_2)$ is similar. \square

Now let us try to dualize the discussion above.

The dual basis B^* is a positive basis of the dual Hopf algebra H^* . In particular, we have $\epsilon = \lambda_1 e_1^* + \cdots + \lambda_l e_l^*$ with $\lambda_i > 0$. Clearly, b^* appears in the sum if and only if $\epsilon(b) > 0$. Since $d_e \in G_+^*$ is the only element in G_+^* with positive ϵ -value, the intersection between $\{e_1, \dots, e_l\}$ and G_+^* is exactly d_e . Moreover, by $1 = \epsilon(1) = \epsilon(d_e)$, we see that the coefficient of d_e^* in the summation for ϵ is 1. Therefore we may rescale e_i to get $\epsilon = e_1^* + \cdots + e_l^*$, without affecting G_+^* .

As in Lemma 2, $\mathbf{C}e_1^* + \cdots + \mathbf{C}e_l^*$ is a commutative Hopf subalgebra of H^* . Therefore $\{e_1, \dots, e_l\}$ form a group G_- under the multiplication in H , with d_e as the unit. To indicate the group structure, we denote

$$\{e_1, \dots, e_l\} = G_- = \{e_{g_-} : g_- \in G_-\}.$$

Then

$$\begin{aligned} e_{g_-} e_{h_-} &= e_{g_- h_-}, \quad \Delta(e_{g_-}) = e_{g_-} \otimes e_{g_-} + \text{terms not in } G_- \otimes G_-, \\ S(e_{g_-}) &= e_{g_-^{-1}} + \text{terms not in } G_-, \quad \epsilon = \sum_{g_- \in G_-} e_{g_-}^*. \end{aligned}$$

Since $d_e \in G_+^*$ is also the identity of G_- , we will abuse the notation and also denote $e = d_e$.

The dual of Lemma 3 is the following: For any $b^* \in B^*$, we have unique $g_-, \bar{g}_- \in G_-$, such that

$$e_{h_-}^* b^* = \delta_{\bar{g}_-, h_-} b^*, \quad b^* e_{h_-}^* = \delta_{g_-, h_-} b^*. \quad (11)$$

This enables us to define

$$\alpha_-(b) = \bar{g}_-, \quad \beta_-(b) = g_-.$$

We may reinterpret (11) as the following lemma. We note that the formula (12), being equivalent to (11), also serves as the characterizations of α_-, β_- .

Lemma 5 Suppose $\alpha_-(b) = \bar{g}_-$ and $\beta_-(b) = g_-$. Then

$$\Delta(b) = e_{\bar{g}_-} \otimes b + b \otimes e_{g_-} + \text{terms not in } B \otimes G_- \text{ or } G_- \otimes B. \quad (12)$$

Proof: By Lemma 1, the equalities $e_{h_-}^* c^* = \delta_{\alpha_-(c), h_-} c^*$ mean that $\Delta(b)$ contains $e_{h_-} \otimes c$ if and only if $c = b$ and $h_- = \alpha_-(b)$. Moreover, when the containment happens, the coefficient is 1. This means exactly that the only $G_- \otimes B$ -term of $\Delta(b)$ is $e_{\alpha_-(b)} \otimes b$, and the coefficient is 1. Similarly, the only $B \otimes G_-$ -term of $\Delta(b)$ is $b \otimes e_{\beta_-(b)}$, also with coefficient 1. □

With the help of Lemma 1, we may also dualize Lemma 4.

Lemma 6 If $b_1 \otimes b_2$ is contained in $\Delta(b)$, then

$$\alpha_-(b_1) = \alpha_-(b), \quad \beta_-(b_2) = \beta_-(b). \quad (13)$$

If $b_1 b_2$ contains b , then

$$\alpha_-(b) = \alpha_-(b_1) \alpha_-(b_2), \quad \beta_-(b) = \beta_-(b_1) \beta_-(b_2). \quad (14)$$

□

Having found G_+ and G_- , we need to further construct $G = G_+ G_-$. One approach is to make use of the Drinfel'd double $D(H)$, which has $B^* \otimes B$ as a positive basis. From $D(H)$ we can also construct a “positive group” \tilde{G}_+ and a “negative group” \tilde{G}_- . \tilde{G}_- consists of exactly the pairs $b_1^* \otimes b_2 \in B^* \otimes B$, such that

$$\epsilon(b_1^* \otimes b_2) = \langle b_1, 1 \rangle \epsilon(b_2) > 0$$

Thus $\tilde{G}_- = G_+ \otimes G_-$ and this is a unique factorization. It remains to verify that $H(\tilde{G}_-; G_+, G_-)$ is isomorphic to the Hopf algebra H .

Instead of making use of Drinfel'd double, we will continue with a more elementary approach. The approach will also be applicable to the Hopf algebras in the correspondence category, where the notion of Drinfel'd double is yet to be defined.

Lemma 7 The following are equivalent

- (1) $\alpha_+(b) = e$;
- (2) $\beta_+(b) = e$;
- (3) $b \in G_-$.

In particular, $B_{e, g_+} = B_{g_+, e} = \emptyset$ for $g_+ \neq e$.

Proof: Since $d_e = e$ is also the multiplicative unit for G_- , we see (3) implies (1). Conversely, from the last lemma, $\Delta(b)$ contains $e_{\bar{g}_-} \otimes b$. Then by $\epsilon = m(S \otimes 1)\Delta$, $\epsilon(b)$ contains $S(e_{\bar{g}_-})b$, which further contains $e_{\bar{g}_-^{-1}}b$. If b satisfies (1), then $e_{\bar{g}_-}\epsilon(b)$ contains $e_{\bar{g}_-}e_{\bar{g}_-^{-1}}b = d_e b = b$, so that $\epsilon(b) \neq 0$. This implies $b \in G_-$. Similarly, by using $\epsilon = m(1 \otimes S)\Delta$, we can prove (2) implies (3). \square

Lemma 8 *Given $g_+ \in G_+$ and $g_- \in G_-$, there is a unique $B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -term $c \otimes b$ contained in $\Delta(e_{g_-})$. Moreover, b is the unique element such that $\alpha_+(b) = g_+$ and $\beta_-(b) = g_-$.*

Proof: We begin by finding a $B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -term in $\Delta(e_{g_-})$. Since $m(1 \otimes S)\Delta(e_{g_-}) = \epsilon(e_{g_-}) = 1 = \sum_{h_+ \in G_+} d_{h_+}$, $\Delta(e_{g_-})$ must contain a term $c \otimes b$, such that $cS(b) = \sum_{h_+ \in G_+} \lambda_{h_+} d_{h_+}$ with $\lambda_{g_+^{-1}} > 0$. Then we have $d_{g_+^{-1}}cS(b) = \lambda_{g_+^{-1}}d_{g_+^{-1}}$. Thus $d_{g_+^{-1}}c \neq 0$, and by (7), $\alpha_+(c) = g_+^{-1}$. Then by Lemmas 4, 6, and 7, we have $\alpha_+(b) = \alpha_+(c)^{-1}\alpha_+(e_{g_-}) = g_+$.

Next we would like to compare the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $(\Delta \otimes 1)\Delta(b)$ and $(1 \otimes \Delta)\Delta(b)$. For $(\Delta \otimes 1)\Delta(b)$, such terms can only be obtained from applying $\Delta \otimes 1$ to $B \otimes B_{g_+, \bullet}$ -terms in $\Delta(b)$. Now for any $B \otimes B_{g_+, \bullet}$ -term $b_1 \otimes b_2$ contained in $\Delta(b)$, we apply Lemma 4 and find $g_+ = \alpha(b) = \alpha_+(b_1)\alpha_+(b_2) = \alpha_+(b_1)g_+$. Therefore $\alpha_+(b_1) = e$, so that by Lemma 7, we have $b_1 \in G_-$. Thus $b_1 \otimes b_2$ is really a $G_- \otimes B_{g_+, \bullet}$ -term contained in $\Delta(b)$. It then follows from Lemma 5 that $b_1 \otimes b_2 = e_{\bar{g}_-} \otimes b$ (where $\bar{g}_- = \alpha_-(b)$), and the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $(\Delta \otimes 1)\Delta(b)$ come from $\Delta(e_{\bar{g}_-}) \otimes b$. Similarly, the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $(1 \otimes \Delta)\Delta(b)$ come from $b \otimes \Delta(e_{g_-})$.

Let $\sum_i \lambda_i c_i \otimes b_i$ be all the $B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $\Delta(e_{g_-})$. Let $\sum_j \lambda'_j b'_j \otimes c'_j$ be all the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet}$ -terms in $\Delta(e_{\bar{g}_-})$. Then by comparing the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $\Delta(e_{\bar{g}_-}) \otimes b$ and $b \otimes \Delta(e_{g_-})$, we get

$$\sum_i \lambda_i b \otimes c_i \otimes b_i = \sum_j \lambda'_j b'_j \otimes c'_j \otimes b.$$

Consequently, all $b_i = b$, all $b'_j = b$, and

$$\Delta(e_{g_-}) = \left(\sum_i \lambda_i c_i \right) \otimes b + \text{no } B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}\text{-terms} \quad (15)$$

$$\Delta(e_{\bar{g}_-}) = b \otimes \left(\sum_j \lambda'_j c'_j \right) + \text{no } B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet}\text{-terms} \quad (16)$$

The equality (15) shows that for any $g_+ \in G_+$ and $g_- \in G_-$, all the $B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $\Delta(e_{g_-})$ have the same right component. Now apply this fact to $g_+^{-1} \in G_+$ and $\bar{g}_- \in G_-$,

we see that all the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet}$ -terms in $\Delta(e_{\bar{g}_-})$ have the same right component. Then (16) tells us that $\sum_i \lambda_i c_i$ is really only one term. We thus conclude that for any $g_+ \in G_+$ and $g_- \in G_-$, the $B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -term in $\Delta(e_{g_-})$ is unique.

Now let us prove the uniqueness of the element b satisfying $\alpha_+(b) = g_+$ and $\beta_-(b) = g_-$. We already have one such element appearing in

$$\Delta(e_{g_-}) = c \otimes b + \text{no } B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}\text{-terms}$$

Now suppose b' is another base element satisfying $\alpha_+(b') = g_+$ and $\beta_-(b') = g_-$. As before, we write (we do not know apriori that $\bar{g}'_- = \bar{g}_-$)

$$\Delta(b') = e_{\bar{g}'_-} \otimes b' + b' \otimes e_{g_-} + \text{terms not in } B \otimes G_- \text{ or } G_- \otimes B,$$

and compare the $B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -terms in $(\Delta \otimes 1)\Delta(b')$ and $(1 \otimes \Delta)\Delta(b')$. The result is

$$[B_{g_+, \bullet} \otimes B_{g_+^{-1}, \bullet}\text{-term in } \Delta(e_{\bar{g}'_-})] \otimes b' = b' \otimes c \otimes b.$$

Comparing the rightmost components, we conclude that $b = b'$. □

We make some remarks about the lemma above.

First of all, observe that $c \otimes b$ is also the unique $B \otimes B_{g_+, \bullet}$ -term (as well as the unique $B_{g_+^{-1}, \bullet} \otimes B$ -term) in $\Delta(e_{g_-})$. As a matter of fact, by Lemma 4, any $B \otimes B_{g_+, \bullet}$ -term in $\Delta(e_{g_-})$ must be a $B_{g_+^{-1}, \bullet} \otimes B_{g_+, \bullet}$ -term.

Secondly, if we start with $\bar{g}_+ \in G_+$ and $\bar{g}_- \in G_-$, similar proof also tells us that there is a unique $B_{\bullet, \bar{g}_+} \otimes B_{\bullet, \bar{g}_+^{-1}}$ -term $b \otimes c$ contained in $\Delta(e_{\bar{g}_-})$. Moreover, b is the unique element such that $\beta_+(b) = \bar{g}_+$ and $\alpha_-(b) = \bar{g}_-$. Briefly speaking, one may start with $m(S \otimes 1)\Delta(e_{\bar{g}_-}) = 1$ and conclude $\Delta(e_{\bar{g}_-})$ contains a $B_{\bullet, \bar{g}_+} \otimes B_{\bullet, \bar{g}_+^{-1}}$ -term. Then by comparing the $B_{\bullet, \bar{g}_+} \otimes B_{\bullet, \bar{g}_+^{-1}} \otimes B_{\bullet, \bar{g}_+}$ -terms in $(\Delta \otimes 1)\Delta(b)$ and $(1 \otimes \Delta)\Delta(b)$, our claim follows. In particular, we also see that $b \rightarrow (\beta_+(b), \alpha_-(b))$ is a one-to-one correspondence.

Thirdly, we have the one-to-one correspondences

$$(g_+, g_-) \xrightarrow{(\alpha_+, \beta_-)} b \xrightarrow{(\beta_+, \alpha_-)} (\bar{g}_+, \bar{g}_-), \quad (17)$$

and the equalities

$$\Delta(e_{g_-}) = \lambda c \otimes b + \dots \quad (18)$$

$$\Delta(e_{\bar{g}_-}) = \lambda b \otimes c + \dots \quad (19)$$

The one-to-one correspondences (17) enables us to define a right action of G_- on G_+

$$G_+ \times G_- \rightarrow G_+ : (g_+, g_-) \mapsto g_+^{g_-} = \bar{g}_+, \quad (20)$$

and a left action of G_+ on G_-

$$G_+ \times G_- \rightarrow G_- : (g_+, g_-) \mapsto {}^{g_+}g_- = \bar{g}_-. \quad (21)$$

These are presumably the actions (4) and (5) in our standard model. In Lemma 12, we will show they are indeed actions and satisfy matching relations (6).

The action (20) has the following interpretation: Given g_+ and g_- , we use (18) to find the unique $b \in B_{g_+, \bullet}$. Then $\bar{g}_+ = \beta_+(b)$ is ${}^{g_-}g_+$ (i.e., $b \in B_{g_+, \bar{g}_+}$).

The action (21) has the following interpretation: Given g_+ and g_- , we establish (18) and (19) with $b \in B_{g_+, \bullet}$. Then the element \bar{g}_- in (19) is $g_-^{g_+}$.

The reason for the interpretations above follows from (15), (16), and the uniqueness. We may also find g_+ , g_- from (\bar{g}_+, \bar{g}_-) in the similar way. These are the other two actions in the standard model, and we will not use them.

Lemma 9 *For any $b \in B_{g_+, \bar{g}_+}$, there is $b' \in B_{\bar{g}_+, g_+}$, such that $bb' = \lambda d_{g_+}$ and $b'b = \lambda d_{\bar{g}_+}$ for some $\lambda > 0$.*

Proof: Let $\beta_-(b) = g_-$. From Lemma 8 and the first two of the subsequent remarks, we have a unique $B \otimes B_{\bullet, \bar{g}_+}$ -term $c \otimes b$ in $\Delta(e_{g_-})$. Then by $m(S \otimes 1)\Delta(e_{g_-}) = \epsilon(e_{g_-}) = 1$, we have $S(c)b = \mu d_{\bar{g}_+}$ for some $\mu > 0$. By positivity, there is some term b' in $S(c)$, such that $b'b = \lambda d_{\bar{g}_+}$ for some $\lambda > 0$.

We claim that $b' \in B_{\bar{g}_+, g_+}$. From $d_{\bar{g}_+}b'b = \lambda d_{\bar{g}_+}^2 = \lambda d_{\bar{g}_+} \neq 0$, we have $d_{\bar{g}_+}b' \neq 0$, so that $\alpha_+(b') = g_+$. From $(b'd_{g_+})b = b'(d_{g_+}b) = b'b = \lambda d_{\bar{g}_+} \neq 0$, we have $b'd_{g_+} \neq 0$, so that $\beta_+(b') = \bar{g}_+$.

We have shown that for any $b \in B_{g_+, \bar{g}_+}$, there is $b' \in B_{\bar{g}_+, g_+}$, such that $b'b = \lambda d_{\bar{g}_+}$ for some $\lambda > 0$. Applying this conclusion to b' , there is $b'' \in B_{g_+, \bar{g}_+}$, such that $b''b' = \lambda' d_{g_+}$ for some $\lambda' > 0$. Then we have

$$b''b'b = (b''b')b = \lambda' d_{g_+}b = \lambda'b, \quad b''b'b = b''(b'b) = \lambda b'' d_{\bar{g}_+} = \lambda b''.$$

This implies that $\lambda = \lambda'$ and $b'' = b$. □

The next lemma shows that the antipode S is a permutation on B up to rescaling. We give a proof that is also valid in the correspondence category. For Hopf algebras over \mathbf{C} , it can also be proved more directly by using the positivity of S and the fact that $S^N = id$ for some positive integer N .

Lemma 10 *Given any $b \in B$, $S(b) = \lambda b'$ for some $b' \in B$ and $\lambda > 0$. Moreover, we have $\alpha_-(b') = \beta_-(b)^{-1}$, $\beta_+(b') = \alpha_+(b)^{-1}$, and these uniquely determine b' .*

Proof: First we show that $S(e_{g_-}) = e_{g_-^{-1}}$. We already know $S(e_{g_-}) = e_{g_-^{-1}} + \text{terms not in } G_-$. Suppose $g_- \neq e$, and $S(e_{g_-})$ contains $b \notin G_-$. Then by Lemma 7, $h_+ = \alpha_+(b) \neq e$, so that $d_{h_+}S(e_{g_-}) = S(e_{g_-}d_{h_+^{-1}}) = 0$. On the other hand, $d_{h_+}S(e_{g_-})$ still contains $d_{h_+}b = b$ and cannot vanish. The contradiction shows that $S(e_{g_-}) = e_{g_-^{-1}}$. On the other hand, suppose $g_- = e$. Then we already know $e_{g_-} = d_e$ and $S(d_e) = d_e$.

Let g_{\pm} and \bar{g}_{\pm} be associated to b as usual. Then from Lemma 8 we have

$$\Delta(e_{g_-}) = \lambda c \otimes b + \dots$$

Applying S , we have

$$\Delta(e_{g_-^{-1}}) = \lambda S(b) \otimes S(c) + \dots$$

The proof of the last lemma implies that $S(c)$ contains a term $c' \in B_{g_+, \bar{g}_+}$. Therefore if $S(b)$ contains b' , then $b' \otimes c'$ must be contained in $\Delta(e_{g_-^{-1}})$. By the first remark after Lemma 8, the $B \otimes B_{g_+, \bar{g}_+}$ -term in $\Delta(e_{g_-^{-1}})$ must be unique. Therefore for the fixed c' , there is only one term like $b' \otimes c'$ in $\Delta(e_{g_-^{-1}})$. This implies that there is only one term, a constant multiple of b' , contained in $S(b)$.

Thus we conclude the first part of the lemma, and have

$$S(b) = \mu b', \quad S(c) = \nu c', \quad \Delta(e_{g_-^{-1}}) = \lambda \mu \nu b' \otimes c' + \dots$$

Moreover, we have also assumed $c' \in B_{g_+, \bar{g}_+}$. By lemmas 4 and 6, we then conclude that $\alpha_-(b') = \beta_-(b)^{-1}$ and $\beta_+(b') = \alpha_+(b)^{-1}$. By the second remark after Lemma 8, this uniquely determines b' . □

Lemma 11 *For $b \in B_{g_+, \bar{g}_+}$ and $c \in B_{h_+, \bar{h}_+}$, there are two possibilities for bc :*

- (1) *If $\bar{g}_+ = h_+$, then bc is a positive multiple of an element in B_{g_+, \bar{h}_+} ;*
- (2) *If $\bar{g}_+ \neq h_+$, then $bc = 0$.*

Proof: If $\bar{g}_+ \neq h_+$, then $bc = (bd_{\bar{g}_+})(d_{h_+}c) = b(d_{\bar{g}_+}d_{h_+})c = 0$.

If $\bar{g}_+ = h_+$, then Lemma 9 immediately implies $bc \neq 0$. Suppose bc contains two different terms $d, d' \in B$. Then by $d_{g_+}bc = bc = bcd_{\bar{h}_+}$ and positivity, we see that $d, d' \in B_{g_+, \bar{g}_+}$. By Lemma 9, there is $b' \in B_{\bar{g}_+, g_+}$, such that $bb' = \lambda d_{g_+}$ and $b'b = \lambda d_{\bar{g}_+}$ for some $\lambda > 0$. Then $b'bc = \lambda c$ contains $b'd$ and $b'd'$. By positivity, we must have $b'd = \mu c$ and $b'd' = \mu' c$ for some $\mu > 0$ and $\mu' > 0$. Thus we conclude that

$$\lambda d = \lambda d_{g_+}d = bb'd = \mu bc, \quad \lambda d' = \lambda d_{g_+}d' = bb'd' = \mu' bc.$$

This is in contradiction with the assumption that d and d' are distinct.

□

Lemma 10 implies that S^{-1} exists, and for any $b \in B$, the coordinates of $S^{-1}(b)$ are non-negative. Therefore $(H^*)^{op}$ is also positively based. In particular, all the proofs we have done so far are valid in $(H^*)^{op}$. Note that the two actions (20) and (21) are related by duality in the following way.

$$\begin{array}{ccccc}
H & \Longleftrightarrow & H^* & \Longleftrightarrow & (H^*)^{op} \\
\alpha_+(b) = g_+ & \longleftrightarrow & \alpha_-(b^*) = \bar{g}_- & \longleftrightarrow & \alpha_-^{op}(b^*) = g_- \\
\beta_+(b) = \bar{g}_+ & \longleftrightarrow & \beta_-(b^*) = g_- & \longleftrightarrow & \beta_-^{op}(b^*) = \bar{g}_- \\
\alpha_-(b) = \bar{g}_- & \longleftrightarrow & \alpha_+(b^*) = g_+ & \longleftrightarrow & \alpha_+^{op}(b^*) = \bar{g}_+ \\
\beta_-(b) = g_- & \longleftrightarrow & \beta_+(b^*) = \bar{g}_+ & \longleftrightarrow & \beta_+^{op}(b^*) = g_+ \\
\bar{g}_+ = g_+^{g_-} & \longleftrightarrow & g_- = \bar{g}_-^{\bar{g}_+} & \longleftrightarrow & \bar{g}_- = g_+^{g_-} \\
\bar{g}_- = g_+^{g_-} & \longleftrightarrow & g_+ = \bar{g}_-^{\bar{g}_+} & \longleftrightarrow & \bar{g}_+ = g_+^{g_-}
\end{array}$$

Therefore by restating the results we have proved in $(H^*)^{op}$, we may exchange the two actions in these results.

Lemma 12 (20) and (21) are indeed actions. Moreover, the two actions satisfy the matching relations (6).

Proof: Given $g_+^{g_-} = h_+$ and $h_+^{h_-} = k_+$, we may find (by the third remark after Lemma 8) $b \in B_{g_+, h_+}$ and $b' \in B_{h_+, k_+}$, with $g_- = \beta_-(b)$ and $h_- = \beta_-(b')$. Then by Lemma 11, we have $bb' = \lambda c$, with $\lambda > 0$ and $c \in B_{g_+, k_+}$. By Lemma 6, we have $\beta_-(c) = g_- h_-$. Therefore $g_+^{g_- h_-} = k_+ = h_+^{h_-} = (g_+^{g_-})^{h_-}$. This proves (20) is a right action.

As explained in the remark above, we may exchange the two actions and still get an equality. Applying the principle to $g_+^{g_- h_-} = (g_+^{g_-})^{h_-}$, we get $h_+ g_+ g_- = h_+ (g_+ g_-)$. This proves (21) is a left action.

Now we prove the matching relations. By the third remark after Lemma 8, we interpret the actions $\bar{g}_+ = g_+^{g_-}$ and $\bar{g}_- = g_+^{g_-}$ as the following equalities

$$\begin{cases} \Delta(e_{g_-}) &= \lambda c \otimes b + \dots, \\ \Delta(e_{\bar{g}_-}) &= \lambda b \otimes c + \dots, \end{cases} \quad b \in B_{g_+, \bar{g}_+}. \quad (22)$$

We also interpret the action $\bar{h}_- = \bar{g}_+ h_-$ as the following equalities

$$\begin{cases} \Delta(e_{h_-}) &= \lambda' c' \otimes b' + \dots, \\ \Delta(e_{\bar{h}_-}) &= \lambda' b' \otimes c' + \dots, \end{cases} \quad b \in B_{\bar{g}_+, \bullet}. \quad (23)$$

Multiplying (22) and (23) together, we have (where we use Lemma 11 so that cc' and bb' are elements of B up positive rescaling)

$$\begin{cases} \Delta(e_{g_- h_-}) = \Delta(e_{g_-}) \Delta(e_{h_-}) &= \lambda \lambda' cc' \otimes bb' + \dots, \\ \Delta(e_{\bar{g}_- \bar{h}_-}) = \Delta(e_{\bar{g}_-}) \Delta(e_{\bar{h}_-}) &= \lambda \lambda' bb' \otimes cc' + \dots, \end{cases} \quad bb' \in B_{g_+, \bullet}. \quad (24)$$

By the third remark after Lemma 8, (24) means $\bar{g}_- \bar{h}_- = {}^{g_+}(g_- h_-)$. Therefore

$${}^{g_+}(g_- h_-) = \bar{g}_- \bar{h}_- = {}^{g_+}g_- \bar{g}_+ h_- = {}^{g_+}g_- (g_+^{g_-}) h_-.$$

This is the matching relation. Applying remark made before the lemma, we get the other matching relation

$$(h_+ g_+)^{g_-} = h_+^{(g_+ g_-)} g_+^{g_-}.$$

□

With a matching group pair G_+, G_- , we may construct a group $G = G_+ G_-$ and projections $\alpha_\pm, \beta_\pm : G \rightarrow G_\pm$. Furthermore, we have a standard Hopf algebra $H(G; G_+, G_-)$ with G as a positive basis. The next Lemma shows that up to positive scaling, the product between the basis elements of H is the same as the product between the basis elements of $H(G; G_+, G_-)$.

Lemma 13 *bb' is a positive multiple of c if and only if*

$$\alpha_+(b) = \alpha_+(c), \quad \beta_+(b) = \alpha_+(b'), \quad \alpha_-(b)\alpha_-(b') = \alpha_-(c).$$

Moreover, for fixed c , $(b, b') \mapsto \alpha_-(b)$ is a one-to-one correspondence between the pairs (b, b') and G_- .

Proof: The necessity follows from (9), (14), and Lemma 11. For sufficiency, the condition $\beta_+(b) = \alpha_+(b')$ implies $bb' = \lambda c'$ for some $\lambda > 0$ and $c' \in B$. Then we have $\alpha_+(c) = \alpha_+(b) = \alpha_+(c')$ and $\alpha_-(c) = \alpha_-(b)\alpha_-(b') = \alpha_-(c')$. If we can show $(\alpha_+(c), \alpha_-(c))$ are in one-to-one correspondence with $(\alpha_+(c), \beta_-(c))$, then it follows from Lemma 8 that $c = c'$.

By the definition of the action of G_+ on G_- , from $(\alpha_+(c), \beta_-(c))$ we have $\alpha_-(c) = \alpha_+(c)\beta_-(c)$. Since this is a group action, from $(\alpha_+(c), \beta_-(c))$ we also have $\beta_-(c) = \alpha_+(c)^{-1}\alpha_-(c)$. This proves the sufficiency.

For the one-to-one correspondence, we fix $\alpha_-(b) \in G_-$. The condition $\alpha_+(b) = \alpha_+(c)$ then uniquely determines b . Moreover, the conditions $\beta_+(b) = \alpha_+(b')$, $\alpha_-(b)\alpha_-(b') = \alpha_-(c)$ give us $\alpha_+(b')$ and $\alpha_-(b')$, which also has a unique corresponding b' . This proves the one-to-one correspondence between pairs (b, b') and G_- .

□

By making use of Lemma 1 and the remark before Lemma 12, we get the following dual of Lemma 13. It shows that up to positive scaling, the coproduct of basis elements of H is the same as the coproduct of basis elements of $H(G; G_+, G_-)$.

Lemma 14 *For any $b \in B$, $\Delta(b)$ contains exactly those $c \otimes c'$ satisfying*

$$\alpha_-(c) = \alpha_-(b), \quad \beta_-(c) = \alpha_-(c'), \quad \alpha_+(c)\alpha_+(c') = \alpha_+(b).$$

Moreover, for fixed b , $c \otimes c' \mapsto \alpha_+(c)$ is a one-to-one correspondence between $c \otimes c'$ and G_+ .

□

4 Rescaling

Given a finite dimensional Hopf algebra H with a positive basis B , we have identified H with a standard Hopf algebra $H(G; G_+, G_-)$ (as well as B with G) up to positive scaling. In this section, we try to positively rescale the elements of G , so that the Hopf algebra structure in H and $H(G; G_+, G_-)$ are exactly the same.

The problem may be viewed in the following way. On the vector space $V = \mathbf{C}G$, we have the standard Hopf algebra structure $H(G; G_+, G_-)$. We also have the given Hopf algebra structure H . The problem is to find a function $\tau : G \rightarrow \mathbf{R}_+$, such that $g \mapsto \tau(g)g$ is a Hopf algebra isomorphism from H to $H(G; G_+, G_-)$.

The Hopf algebra structure H is determined by

$$\begin{aligned} \{g\}\{h\} &= \lambda(g, h)\{gh_-\} \text{ in case } \bar{g}_+ = h_+ \\ \Delta\{g\} &= \sum_{h_+k=g, \bar{k}_-=h_-} \mu(h, k)\{h\} \otimes \{k\} \\ S\{g\} &= \nu(g)\{g^{-1}\} \end{aligned} \tag{25}$$

The structure constants $\lambda(g, h)$, $\mu(h, k)$, $\nu(g)$ are positive and defined exactly when the conditions indicated above are satisfied. We assume H is already well-scaled on G_+^* and G_- , i.e., with canonical products, coproducts, and antipodes, $(\mathbf{C}G_+)^*$ is a sub Hopf algebra of H and $(\mathbf{C}G_-)^*$ is a sub Hopf algebras of H^* . Then the Hopf algebra condition means the following compatibility equations (in cases both sides are defined):

$$\begin{aligned} \lambda(g, h)\lambda(gh_-, k) &= \lambda(g, hk_-)\lambda(h, k), \\ \mu(g, h)\mu(g_+h, k) &= \mu(g, h_+k)\mu(h, k), \\ \lambda(h_+k, h'_+k')\mu(hh'_-, kk'_-) &= \lambda(h, h')\lambda(k, k')\mu(h, k)\mu(h', k'), \\ \lambda(h_+, h) &= 1, \\ \lambda(h, \bar{h}_+) &= 1, \\ \mu(h, h_-) &= 1, \\ \mu(\bar{h}_-, h) &= 1, \\ \nu(h)\lambda(h^{-1}, h_+^{-1}\bar{h}_-) &= 1, \\ \nu(h)\lambda(h_+^{-1}\bar{h}_-, h) &= 1. \end{aligned}$$

It is clear from the equations above that for any real number ϵ , λ^ϵ , μ^ϵ , ν^ϵ still satisfy the compatibility equations. Therefore if we replace λ , μ , ν in (25) with λ^ϵ , μ^ϵ , ν^ϵ , we still get a Hopf algebra (which is also well-scaled on G_+^* and G_-), which we denote by H_ϵ . Then we have $H = H_1$ and $H(G; G_+, G_-) = H_0$.

Lemma 15 *The Hopf algebras H_ϵ are semisimple and cosemisimple.*

Proof: Let $\Lambda = \sum_{g_- \in G_-} \{g_-\}$. Because H_ϵ is already well-scaled on G_- , we have $x\Lambda = \epsilon(x)\Lambda$. Therefore Λ is a left integral of the Hopf algebra. Since $\epsilon(\Lambda) = |G_-| \neq 0$, it follows from [LS] that H_ϵ is semi-simple.

Similarly, the integral $\sum_{g_+ \in G_+} \{g_+\}^*$ tells us that H_ϵ^* is also semi-simple. \square

Consider H_0 as a point in the space X of all bialgebra structures on V . We have natural action of the general linear group $GL(V)$ on X . The orbit $GL(V)H_0$ consists of all the bialgebra structures linearly isomorphic to H_0 . According to the Corollary 1.5 and the Theorem 2.1 of [St], the semisimplicity and the cosemisimplicity of H_ϵ implies that the orbit $GL(V)H_0$ is a Zariski open subset of X . Therefore for sufficiently small ϵ , H_ϵ belongs to the orbit $GL(V)H_0$. In particular, for any $\delta > 0$, we can find an $\epsilon > 0$ and a linear transformation T , such that $T(H_\epsilon) = H_0$ and

$$T\{g\} = \sum_{g \in G} \tau_{g,h}\{h\}, \quad |\tau_{g,g} - 1| < \delta, \quad |\tau_{g,h}| < \delta \quad \text{for } h \neq g. \quad (26)$$

Lemma 16 *For sufficiently small ϵ , T preserves $\mathbf{C}G_-$.*

Proof: Consider $T\{g_-\} = \sum_{h \in G} \tau_{g_-,h}\{h\}$. We would like to show $\tau_{g_-,h} = 0$ for $h \notin G_-$. If not, we find a term $\tau_{g_-,k}\{k\}$, $k \notin G_-$, with

$$|\tau_{g_-,k}| = \max\{|\tau_{g_-,h}| : h \notin G_-\}.$$

Since $k \notin G_-$, we have $k_+ \neq e$. Therefore $\{k_+\}\{g_-\} = 0$ in H_ϵ . Since T preserves products, we have $T\{k_+\}T\{g_-\} = 0$. On the other hand, we will show that for sufficiently small ϵ , the coefficient of $\{k\}$ in $T\{k_+\}T\{g_-\}$ cannot be zero. The contradiction proves the Lemma.

Write

$$\begin{aligned} T\{k_+\} &= \tau_{k_+,k_+}\{k_+\} + \sum_{h \neq k_+} \tau_{k_+,h}\{h\} = P + Q; \\ T\{g_-\} &= \tau_{g_-,k}\{k\} + \sum_{h \in G_-} \tau_{g_-,h}\{h\} + \sum_{h \notin G_-, h \neq k} \tau_{g_-,h}\{h\} = L + M + N. \end{aligned}$$

Suppose (26) is satisfied for some small $\delta > 0$. Then we have

1. $PL = \tau_{k_+,k_+}\tau_{g_-,k}\{k\}$, with $|\tau_{k_+,k_+}\tau_{g_-,k}| \geq (1 - \delta)|\tau_{g_-,k}|$;
2. $P(M + N)$ can never produce $\{k\}$;
3. $Q(L + N)$ may produce some $\{k\}$ -terms. However, the total number of product terms is no more than $(\dim H)^2$, and the coefficients are all smaller than $\delta|\tau_{g_-,k}|$. Therefore the k -coefficient of this part is bounded by $\delta|\tau_{g_-,k}|(\dim H)^2$;

4. QM is a sum of G_- -terms, so that it can never produce $\{k\}$.

Now for sufficiently small ϵ , we can find T satisfying (26), with δ small enough so that $(1 - \delta) > \delta(\dim H)^2$. Then we conclude that the $\{k\}$ -term in $T\{k_+\}T\{g_-\}$ is nontrivial. \square

Lemma 17 *For sufficiently small ϵ , $T(\mathbf{C}(G_+ - \{e\})) \subset \mathbf{C}(G - G_-)$.*

Proof: For any $g_+ \in G_+$, $g_+ \neq e$, write

$$T\{g_+\} = \sum_{h_- \in G_-} \tau_{g_+, h_-} \{h_-\} + \sum_{h \notin G_-} \tau_{g_+, h} \{h\} = P + Q.$$

If P is nontrivial, then we have some $k_- \in G_-$, such that

$$|\tau_{g_+, k_-}| = \max\{|\tau_{g_+, h_-}| : h_- \in G_-\}.$$

Since $g_+ \neq e$, we have $\{g_+\}\{e\} = 0$ in H_ϵ , which implies $T\{g_+\}T\{e\} = 0$. On the other hand, we will show that for sufficiently small ϵ , the coefficient of $\{k_-\}$ in $T\{g_+\}T\{e\}$ cannot be zero. The contradiction proves the lemma.

For sufficiently small ϵ , we have

$$T\{e\} = \tau_{e, e} \{e\} + \sum_{h_- \in G_-, h_- \neq e} \tau_{e, h_-} \{h_-\} = L + M$$

from Lemma 16. Suppose (26) is further satisfied for some small $\delta > 0$. Then we have

1. The k_- -term in PL is $\tau_{g_+, k_-} \tau_{e, e} \{k_-\}$, with $|\tau_{g_+, k_-} \tau_{e, e}| > (1 - \delta) |\tau_{g_+, k_-}|$;
2. PM may produce some $\{k\}$ -terms. However, the total number of product terms is no more than $|G_-|^2$, and the coefficients are all smaller than $\delta |\tau_{g_+, k_-}|$. Therefore k -coefficient of this part is bounded by $\delta |\tau_{g_+, k_-}| |G_-|^2$;
3. $Q(L + M)$ contains no G_- -terms, so that it can never produce $\{k_-\}$.

Now for sufficiently small ϵ , we can find T satisfying Lemma 16 and (26), with δ small enough so that $(1 - \delta) > \delta |G_-|^2$. Then we conclude that the $\{k_-\}$ -term in $T\{g_+\}T\{e\}$ is nontrivial. \square

Lemma 18 *For sufficiently small ϵ , T preserves $\mathbf{C}(G - G_-)$.*

Proof: Let $g \notin G_-$. Then $g_+ \neq e$, so that by Lemma 17, $T\{g_+\}$ contains no G_- -terms.

In H_ϵ , we have $\{g_+\}\{g\} = \{g\}$, which implies $T\{g\} = T\{g_+\}T\{g\}$. Since $T\{g_+\}$ contains no G_- -terms, $T\{g_+\}T\{g\}$ also contains no G_- -terms. Consequently, $T\{g\}$ contains no G_- -terms. \square

The Lemmas 16 and 18 shows that if we decompose $\mathbf{C}G$ into $\mathbf{C}G_- \oplus \mathbf{C}(G - G_-)$, then $T : H_\epsilon \rightarrow H_0$ takes the form

$$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{matrix} \leftarrow G_- \\ \leftarrow G - G_- \end{matrix}. \quad (27)$$

Now we apply this to the dual $(T^*)^{-1} : H_\epsilon^* \rightarrow H_0^*$. Of course to do so may requires us to choose even smaller ϵ , so that all the lemmas work for $(T^*)^{-1}$. Since the groups G_+ and G_- derived from H^* are the same as the groups G_- and G_+ derived from H , we conclude that with regard to the decomposition $(\mathbf{C}G)^* = (\mathbf{C}G_+)^* \oplus (\mathbf{C}(G - G_+))^*$,

$$(T^*)^{-1} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{matrix} \leftarrow G_+^* \\ \leftarrow G^* - G_+^* \end{matrix}.$$

Consequently, we get

$$T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{matrix} \leftarrow G_+ \\ \leftarrow G - G_+ \end{matrix}.$$

In particular, T preserves $\mathbf{C}G_+$.

Lemma 19 *For sufficiently small ϵ , T is identity on G_+ .*

Proof: Let $g_+ \in G_+$. For sufficiently small ϵ , from the discussion above we have

$$T\{g_+\} = \sum_{h_+ \in G_+} \tau_{g_+, h_+} \{h_+\},$$

where τ_{g_+, g_+} is close to 1 and τ_{g_+, h_+} is small for $h_+ \neq g_+$. Moreover, since H_ϵ is already well-scaled on G_+^* , we have $\{g_+\}^2 = \{g_+\}$ in H_ϵ . Therefore in H_0 , we have

$$\sum_{h_+ \in G_+} \tau_{g_+, h_+} \{h_+\} = T\{g_+\} = (T\{g_+\})^2 = \sum_{h_+ \in G_+} \tau_{g_+, h_+}^2 \{h_+\}.$$

Comparing the two sides, we have $\tau_{g_+, h_+} = \tau_{g_+, h_+}^2$. Since τ_{g_+, g_+} is the only coefficient close to 1, we conclude that $\tau_{g_+, h_+} = \delta_{g_+, h_+}$. \square

The lemma also applies to the dual Hopf algebras. Taking the adjoint and then the inverse of (27), we have

$$(T^*)^{-1} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{matrix} \leftarrow G_-^* \\ \leftarrow G^* - G_-^* \end{matrix}. \quad (28)$$

Since G_-^* in H^* plays the role of G_+ in H , the dual of Lemma 19 then shows that $(T^*)^{-1}$ is identity on G_-^* , so that

$$(T^*)^{-1} = \begin{pmatrix} I & 0 \\ 0 & * \end{pmatrix} \begin{matrix} \leftarrow G_-^* \\ \leftarrow G^* - G_-^* \end{matrix}. \quad (29)$$

By taking the inverse and then the adjoint of (29), we see that T is also identity on G_- .

Having proved T is identity on G_\pm , we are ready to show T is a scaling everywhere.

Lemma 20 *For sufficiently small ϵ , T is a scaling on G .*

Proof: For any $g = g_+g_- \in G$, we have $\{g_+\}\{g\} = \{g\}$ in H_ϵ and $T\{g_+\} = \{g_+\}$. Then $T\{g\} = T(\{g_+\}\{g\}) = \{g_+\}T\{g\}$ implies that $T\{g\}$ is a linear combination of terms $\{h\}$ with $h_+ = g_+$. This is equivalent to the fact that, with regard to the disjoint union decomposition $\coprod_{g_+ \in G_+} \{h \in G : \alpha_+(h) = g_+\}$, the matrix of T is a block diagonal one.

Similarly, the matrix of $(T^*)^{-1}$ is a block diagonal one with regard to the decomposition $\coprod_{g_- \in G_-} \{h^* \in G^* : \beta_-(h) = g_-\}$. Therefore the matrix of T is also block diagonal with regard to the decomposition $\coprod_{g_- \in G_-} \{h \in G : \beta_-(h) = g_-\}$. This is equivalent to $T\{g\}$ is a linear combination of terms $\{h\}$ with $h_- = g_-$.

Thus we conclude that $T\{g\}$ is a linear combination of terms $\{h\}$ with $h_+ = g_+$ and $h_- = g_-$. There is only one such term, namely $\{g\}$. Therefore $T\{g\}$ is a scalar multiple of $\{g\}$. □

The fact that $g \mapsto \tau(g)g$ is a Hopf algebra isomorphism from H_ϵ to H_0 means exactly the following equalities

$$\begin{aligned} \lambda(g, h)^\epsilon &= \frac{\tau(g)\tau(h)}{\tau(gh_-)} && \text{in case } \bar{g}_+ = h_+ \\ \mu(g, h)^\epsilon &= \frac{\tau(g_+h)}{\tau(g)\tau(h)} && \text{in case } g_- = \bar{h}_- \\ \nu(g)^\epsilon &= \frac{\tau(g)}{\tau(g^{-1})} \end{aligned}$$

The scaling function in Lemma 20 provides us with a function τ satisfying such equalities. Since all the structure constants λ, μ, ν are positive, this implies that the positive scaling

$g \mapsto |\tau(g)|^{\frac{1}{\epsilon}}$ satisfies the similar equalities without ϵ power. In other words,

$$g \mapsto |\tau(g)|^{\frac{1}{\epsilon}} g : H_1 \rightarrow H_0$$

is an isomorphism of Hopf algebras.

5 Hopf Algebra in Correspondence Category

In this section, we study the set-theoretical version of Theorem 1. We will assume all the sets are finite.

As pointed out in the introduction, if we restrict ourselves to the usual category of finite sets, with maps as morphisms, then we will end up with only finite groups. Therefore we enlarge the category to include correspondences, so that we may get a more interesting class of Hopf algebras.

A *correspondence* from a set X to a set Y is a subset F of $X \times Y$. Given a correspondence F from X to Y and a correspondence from Y to Z , we have the composition

$$F \circ G = \{(x, z) : \text{there is } y \in Y, \text{ such that } (x, y) \in F \text{ and } (y, z) \in G\}.$$

With correspondences as morphisms, the finite sets form a category, which we call the *correspondence category*. With the usual product of sets as tensor product, a one point set as an identity object, and the diagonal $I_X = \{(x, x) : x \in X\} \subset X \times X$ as the identity morphism, the correspondences category becomes a *monoidal category* as defined in [Mac].

One may introduce the concept of algebra (called monoid in [Mac]) in any monoidal category. By reversing the directions of arrows in all diagrams used in defining algebra, we have the concept of coalgebra. In the correspondence category, the product of two algebras is also an algebra. A *correspondence bialgebra* consists of an algebra structure and a coalgebra structure, such that the coalgebra map is an algebra morphism. A *correspondence Hopf algebra* is a correspondence bialgebra with an antipode making the usual diagrams commutative.

The purpose of this section is to classify correspondence Hopf algebras.

It is not convenient to work with correspondences directly. So we take the following Boolean viewpoint. First we observe that a correspondence $F \subset X \times Y$ from X to Y gives rise to a map on the power sets

$$P_F : P(X) \rightarrow P(Y),$$

$$P_F(A) = \{b \in Y : \text{There is } a \in A, \text{ such that } (a, b) \in F\}.$$

The map P_F preserves union and sends empty set to empty set. Conversely, if a map $P : P(X) \rightarrow P(Y)$ preserves union and sends empty set to empty set, then we have $P = P_F$ for the unique F given by $F = \{(a, b) : b \in P(\{a\})\}$.

Power sets can be considered as modules over the simplest *Boolean algebra* \mathbf{B} . \mathbf{B} consists of two elements 0 and 1, and has the commutative sum and product

$$0 + 0 = 0, \quad 1 + 0 = 1 + 1 = 1, \quad 0 \cdot 0 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

\mathbf{B} satisfies all the axioms of a ring except the existence of additive inverse. A \mathbf{B} -module is a set M with addition $+$, a special element 0, and a scalar multiplication $\mathbf{B} \times M \rightarrow M$, such that

1. $+$ is commutative, associative, and $0 + x = x$ for all $x \in M$;
2. $1 \cdot x = x$, $0 \cdot x = 0$;
3. $(b_1 + b_2) \cdot x = b_1 \cdot x + b_2 \cdot x$, $b \cdot (x_1 + x_2) = b \cdot x_1 + b \cdot x_2$.

It is straightforward to check that, under the first two conditions, the third condition means exactly

$$x + x = x, \quad \text{for all } x \in M. \quad (30)$$

In fact, a \mathbf{B} -module is equivalent to a lattice. In view of our assumption that all sets are finite, we will also assume all \mathbf{B} -modules are finite.

With the obvious notion of homomorphisms, (finite) \mathbf{B} -modules form a category. The power set $P(X)$ is a \mathbf{B} -module by

$$A + B = A \cup B, \quad 0 = \emptyset.$$

It is easy to see that

$$P : X \mapsto P(X), F \mapsto P_F$$

is a functor from the correspondence category to the \mathbf{B} -module category.

A subset B of a \mathbf{B} -module M is called a *basis* if every $x \in M$ is a unique linear combination of elements in B . A \mathbf{B} -module is called *free* if it has a basis. Clearly, the collection $\{\{x\} : x \in X\}$ is a basis of $P(X)$. The following lemma (especially the third property) implies P is a one-to-one correspondence between finite sets and finite free \mathbf{B} -modules.

Lemma 21 *Let B be a basis of a \mathbf{B} -module M . Let $b \in B$.*

1. *If the b -coordinate of $x \in M$ is 1, then for any $y \in M$, the b -coordinate of $x + y$ is also 1;*
2. *If $b = x_1 + \cdots + x_k$ for some $x_i \in M$, then each $x_i = b$ or 0;*
3. *The basis of M is unique.*

Proof: In the first property, we have $x = b + \xi$, where ξ is a sum of elements in $B - \{b\}$. We also have two possibilities for y :

$$y = b + \eta, \quad \text{or} \quad y = \eta,$$

where η is a sum of elements in $B - \{b\}$. Then by (30), we have

$$x + y = \begin{cases} b + b + \xi + \eta & \text{if } y = b + \eta \\ b + \xi + \eta & \text{if } y = \eta \end{cases} = b + \xi + \eta,$$

and $\xi + \eta$ is still a sum of elements in $B - \{b\}$. This proves the first property.

In the second property, if x_1 is neither b nor 0 , then we have $b' \in B - \{b\}$, such that the b' -coordinate of x_1 is 1 . By taking $x = x_1$, $y = x_2 + \cdots + x_k$, and $b = b'$ in the first property, we see that the b' -coordinate of b is also 1 . The contradiction shows that $x_1 = b$ or 0 . The proof for x_i is similar.

If we take the expression $b = x_1 + \cdots + x_k$ in the second property as the expansion of b (in one basis B) in terms of another basis (consisting of x_1, \dots, x_k and some others), then we immediately see that B is the only basis of M . □

The tensor product of \mathbf{B} -modules can be defined in the same way as modules over commutative rings. With \mathbf{B} as the unit object, \mathbf{B} -modules then form a monoidal category. Since we clearly have $P(X) \otimes P(Y) = P(X \times Y)$ and $P(\text{one point}) = \mathbf{B}$, P is a functor of monoidal categories.

The concepts such as \mathbf{B} -algebra, \mathbf{B} -coalgebra, and \mathbf{B} -Hopf algebra can be introduced just like the corresponding concepts over commutative rings. Since P is a functor of monoidal categories, P sends correspondence Hopf algebras to free \mathbf{B} -Hopf algebras. Moreover, because of the third property in Lemma 21, P is a one-to-one correspondence between correspondence Hopf algebras and free \mathbf{B} -Hopf algebras.

Motivated by the classification of positively based Hopf algebras over \mathbf{C} , we may introduce free \mathbf{B} -Hopf algebras $H(G; G_+, G_-)$ as in Section 2 and conjecture that these are all the free \mathbf{B} -Hopf algebras. This is indeed the case.

Theorem 2 *Any free \mathbf{B} -Hopf algebra is isomorphic to $H(G; G_+, G_-)$ for a unique group G and a unique factorization $G = G_+ G_-$.*

Proof: The theorem may be proved by basically repeating the contents of Section 2. There is no need to do rescaling (and all the positive coefficients in Section 2 can be replaced by 1) since the only nonzero element of \mathbf{B} is 1 .

Instead of repeating all the details, we discuss some key points involved in Section 2, and prove only one lemma to illustrate that the whole section works in the category of \mathbf{B} -modules.

There are two key technical points in the proofs in Section 2: One is the use of duality, opposite, and co-opposite in translating a proven lemma into another lemma in similar or dual settings. The other is the use of positivity.

The dual of \mathbf{B} -modules can be defined in the same way as modules over commutative rings, and satisfies similar properties such as $(M^*)^* = M$. The opposite and co-opposite of \mathbf{B} -Hopf algebras can be introduced similarly, as far as the antipode is invertible.

As for the positivity, the role can be replaced by the three properties in Lemma 30. Here we only illustrates how this works by proving Lemma 2. Assume B is the basis of a \mathbf{B} -Hopf algebra H . The identity element 1 of H is a summation $1 = d_1 + \cdots + d_k$ of basis elements. Then we have $d_i = d_i \cdots 1 = d_i d_1 + \cdots + d_i d_k$. Because d_i is a basis element, by the second property in Lemma 30 we see that $d_i d_j = 0$ or d_i . Similarly, from $d_j = 1 \cdot d_j$, we see that $d_i d_j = 0$ or d_j . Then we may conclude $d_i d_j = 0$ when $i \neq j$ and $d_i d_i = d_i$, as before.

Similarly, from $\Delta(d_1) + \cdots + \Delta(d_k) = \Delta(1) = 1 \otimes 1 = \sum_{i,j} d_i \otimes d_j$, we may prove $\mathbf{B}d_1 + \cdots + \mathbf{B}d_k$ is closed under Δ . From $S(d_1) + \cdots + S(d_k) = S(1) = 1 = d_1 + \cdots + d_k$, we may prove $\mathbf{B}d_1 + \cdots + \mathbf{B}d_k$ is closed under S . Therefore $\mathbf{B}d_1 + \cdots + \mathbf{B}d_k$ is a Hopf subalgebra of H . The standard argument for classifying commutative Hopf algebras also applies to \mathbf{B} -modules. Then we know $\mathbf{B}d_1 + \cdots + \mathbf{B}d_k$ must be the dual \mathbf{B} -Hopf algebra of the group algebra $\mathbf{B}G_+$. This completes the proof of the Lemma 2 for \mathbf{B} -Hopf algebras. \square

By the one-to-one correspondence between correspondence Hopf algebras and free \mathbf{B} -Hopf algebras, we may translate Theorem 2 into the following classification of correspondence Hopf algebras.

Theorem 3 *Any finite correspondence Hopf Algebra is a finite group G , with a unique factorization $G = G_+ G_-$, such that*

$$\begin{aligned} m &= \{(g_- h_+, h_+ k_-, g_- h_+ k_-) : h_+ \in G_+, g_-, k_- \in G_-\} \subset (G \times G) \times G \\ \Delta &= \{(g_+ h_- k_+, g_+ h_-, h_- k_+) : g_+, k_+ \in G_+, h_- \in G_-\} \subset G \times (G \times G) \\ 1 &= \{(pt, g_+) : g_+ \in G_+\} \subset \{pt\} \times G \\ \epsilon &= \{(g, pt) : g_- \in G_-\} \subset G \times \{pt\} \\ S &= \{(g, g^{-1}) : g \in G\} \subset G \times G \end{aligned}$$

\square

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